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Coloured Black Holes
in Higher Curvature String Gravity

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Abstract

We consider the combined Yang Mills-Dilaton-Gravity system in the presence of a Gauss-Bonnet term as it appears in the $4D$ Effective Superstring Action. We give analytical arguments and demonstrate numerically the existence of black hole solutions with non-trivial dilaton and Yang Mills hair for the particular case of SU(2) gauge fields. The thermodynamical properties of the solutions are also discussed.

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1. Dilaton gravity with a Gauss-Bonnet term and an $SU(2)$ Yang Mills potential.

The effective theory of gravity resulting from String Theory [1] at low energies includes important modifications of Einstein's theory due to the presence of the extra degrees of freedom such as dilatons, axions and Yang Mills fields. The loop-corrected Superstring Effective Action through the contribution of the Gauss-Bonnet term leads to the existence of singularity-free cosmological solutions [2] [3] as well as to the existence of new dilatonic black holes [4] [5]. The black hole solutions found possess hair [6] outside the horizon as anticipated from general and perturbative in α' considerations [7] [8]. The existence of black holes with non-trivial hair has been demonstrated in the Einstein-Maxwell-Dilaton system [9] [10] and the Einstein-Yang Mills(-Dilaton-Axion) system [11]. In the present article we shall extend the analysis of reference [4] by including an SU(2) Yang Mills field. We find new black hole solutions of the combined gravity-dilaton-SU(2) Yang Mills system in the (crucial) presence of the Gauss-Bonnet term.

Let us consider the heterotic string effective action, ignoring for simplicity moduli and axion fields. Following the notation of reference [4], we have

$$\mathcal{L} = -\frac{1}{2}R - \frac{1}{4}(\partial_\mu \phi)^2 + \frac{\alpha'}{8g^2}e^{\phi}(R^2_{GB} - F^{\alpha\mu\nu}F^{\alpha}_{\ \mu\nu}) \tag{1}$$

where α' is the Regge slope, g^2 is some gauge coupling constant and $F_{\mu\nu}$ is the Yang Mills field strength. The Gauss- Bonnet term is defined as

$$R^2_{GB} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \tag{2}$$

We shall consider only the contribution of an SU(2) Yang Mills field, assuming trivial values for all other Yang Mills as well as "matter" fields.

At this point we shall make a spherically symmetric ansatz for the space-time metric

$$ds^2 = -e^{\Gamma}dt^2 + e^{\Lambda}dr^2 + r^2(d\theta^2 + sin^2\theta d\varphi^2) \tag{3}$$

The functions Γ and Λ depend solely on the radius r. For the SU(2) Yang Mills potential we shall adopt the magnetic ansatz [11]

$$\mathbf{A} = \frac{1+w}{e}\left[-\hat{\tau}_\varphi d\theta + \hat{\tau}_\theta sin\theta d\varphi\right] \tag{4}$$

where e is an SU(2) gauge coupling constant, $w = w(r)$ is a radial function, and $(\hat{\tau}_r, \hat{\tau}_\theta, \hat{\tau}_\varphi)$ is the anti-Hermitian SU(2) basis expressed in polar coordinates ; e.g.

$\hat{\tau}_r = \hat{\mathbf{r}} \cdot \tau$, and $[\hat{\tau}_a, \hat{\tau}_b] = \epsilon_{abc}\hat{\tau}_c$ with the indices ranging over (r, θ, φ). The field strength corresponding to (4) is

$$\mathbf{F} = \frac{w'}{e}\left[-\hat{\tau}_\varphi\, dr \wedge d\theta + \hat{\tau}_\theta\, sin\theta\, dr \wedge d\varphi\right] - \frac{(1-w^2)}{e}\,\hat{\tau}_r\, sin\theta\, d\theta \wedge d\varphi \tag{5}$$

2. Some analytical considerations.

Using the static, spherically symmetric ansatz (3) and (5), the dilaton and Yang-Mills equation as well as the (tt), (rr) and $(\theta\theta)$ component of the Einstein's equations resulting from (1) take the form

$$\begin{aligned}
\phi'' + \phi'(\frac{\Gamma' - \Lambda'}{2} + \frac{2}{r}) &= \frac{\alpha' e^{\phi}}{g^2 r^2}\left(\Gamma'\Lambda' e^{-\Lambda} + (1 - e^{-\Lambda})[\Gamma'' + \frac{\Gamma'}{2}(\Gamma' - \Lambda')]\right. \\
&+ \left.\frac{w'^2}{e^2} + \frac{(1-w^2)^2}{2e^2r^2}\, e^{\Lambda}\right) \qquad (6) \\
w'' + w'\,(\frac{\Gamma' - \Lambda'}{2} + \phi') &+ \frac{w(1-w^2)}{r^2}\, e^{\Lambda} = 0 \qquad (7) \\
\Lambda'\left(1 + \frac{\alpha' e^{\phi}}{2g^2 r}\phi'(1 - 3e^{-\Lambda})\right) &= \frac{r\phi'^2}{4} + \frac{1 - e^{\Lambda}}{r} + \frac{\alpha' e^{\phi}}{g^2 r}(\phi'' + \phi'^2)(1 - e^{-\Lambda}) \\
&+ \frac{\alpha' e^{\phi}}{2g^2 r}\left(\frac{w'^2}{e^2} + \frac{(1-w^2)^2}{2e^2 r^2}\, e^{\Lambda}\right) \qquad (8) \\
\Gamma'\left(1 + \frac{\alpha' e^{\phi}}{2g^2 r}\phi'(1 - 3e^{-\Lambda})\right) &= \frac{r\phi'^2}{4} + \frac{e^{\Lambda} - 1}{r} + \frac{\alpha' e^{\phi}}{2g^2 r}\left(\frac{w'^2}{e^2} - \frac{(1-w^2)^2}{2e^2 r^2}\, e^{\Lambda}\right) \qquad (9) \\
\Gamma'' + \frac{\Gamma'}{2}(\Gamma' - \Lambda') + \frac{\Gamma' - \Lambda'}{r} &= -\frac{\phi'^2}{2} + \frac{\alpha' e^{\phi - \Lambda}}{g^2 r}\left(\phi'\Gamma'' + (\phi'' + \phi'^2)\Gamma'\right. \\
&+ \left.\frac{\phi'\Gamma'}{2}(\Gamma' - 3\Lambda')\right) + \frac{\alpha' e^{\phi}}{2g^2 e^2 r^4}\,(1-w^2)^2\, e^{\Lambda} \qquad (10)
\end{aligned}$$

The corresponding components of $T_{\mu\nu}$ are

$$\begin{aligned}
T_t^t &= -e^{-\Lambda}\frac{\phi'^2}{4} - \frac{\alpha' e^{\phi - \Lambda}}{g^2 r^2}\left\{(\phi'' + \phi'^2)(1 - e^{-\Lambda}) - \frac{\phi'\Lambda'}{2}(1 - 3e^{-\Lambda}) + \frac{w'^2}{2e^2} + \frac{(1-w^2)^2 e^{\Lambda}}{4e^2 r^2}\right\} \\
T_r^r &= e^{-\Lambda}\frac{\phi'^2}{4} - \frac{\alpha'}{g^2 r^2}e^{\phi - \Lambda}\left\{\frac{\phi'\Gamma'}{2}(1 - 3e^{-\Lambda}) - \frac{w'^2}{2e^2} + \frac{(1-w^2)^2 e^{\Lambda}}{4e^2 r^2}\right\} \qquad (11) \\
T_\theta^\theta &= -e^{-\Lambda}\frac{\phi'^2}{4} + \frac{\alpha' e^{\phi - 2\Lambda}}{2g^2 r}\left\{\Gamma''\phi' + \Gamma'(\phi'' + \phi'^2) + \frac{\Gamma'\phi'}{2}(\Gamma' - 3\Lambda') + \frac{(1-w^2)^2 e^{2\Lambda}}{2e^2 r^3}\right\}
\end{aligned}$$

Due to the presence of the higher curvature terms, the assumption of positive definiteness of the time-component of the "energy-momentum" tensor breaks down. This point is crucial for the evasion of the no-hair conjecture [6] and the existence of new black hole solutions.

Let us study now our system at infinity. Demanding asymptotically flat solutions, we expand the functions $e^{\Lambda(r)}$, $e^{\Gamma(r)}$, $\phi(r)$ and $w(r)$ in a power series in $1/r$ and substitute them in the equations (6) - (10). From eq.(7) in the first order we take the constraint: $w_\infty (1 - w_\infty^2) = 0$. As a result, the only two options allowed are either $w_\infty = 0$ or $w_\infty = \pm 1$. The first leads to $w_1 = 0$ in the second order, to $w_2 = 0$ in the third order and eventually to the vanishing of all w_n's. This case corresponds to an Abelian potential

$$\mathcal{A} \sim \frac{1/e}{r} \Rightarrow B_r = F_{\theta\varphi} \sim \frac{1/e}{r^2} \tag{12}$$

with magnetic charge $1/e$. The other option, namely $w_\infty = \pm 1$, leads to non-vanishing w_n's and corresponds to the non-Abelian sector of SU(2). In this case the radial magnetic field is $B_r \sim 1/r^3$ implying that the Yang Mills charge Q_{YM} vanishes at infinity. We shall study the non-Abelian case in this article and, thus, make the choice $w_\infty = \pm 1$ which leads us to the following asymptotic form of the solution near infinity

$$e^{\Lambda(r)} = 1 + \frac{2M}{r} + \frac{16M^2 - D}{4r^2} + O(\frac{1}{r^3}) \tag{13}$$

$$e^{\Gamma(r)} = 1 - \frac{2M}{r} + O(\frac{1}{r^3}) \tag{14}$$

$$\phi(r) = \phi_\infty + \frac{D}{r} + \frac{DM}{r^2} + O(\frac{1}{r^3}) \tag{15}$$

$$w(r) = \pm\left(1 + \frac{D_w}{r} + O(\frac{1}{r^2})\right) \tag{16}$$

M is the ADM mass and D is the dilaton charge defined over a two-sphere at infinity as [10]

$$D = -\frac{1}{4\pi}\int d^2\Sigma^\mu \nabla_\mu \phi \tag{17}$$

Let us now direct our attention to the other end of the allowed spatial range, i.e. near the event horizon. We note that equation (9) can be solved for e^Λ as

$$e^\Lambda = \frac{-\beta + \delta\sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} \tag{18}$$

where $\delta = \pm 1$ and

$$\begin{aligned}
\alpha &= 1 - \frac{\alpha' e^{\phi}}{4g^2 r^2} \frac{(1-w^2)^2}{e^2} \quad , \quad \gamma = \frac{3\alpha' e^{\phi}}{2g^2} \Gamma' \phi' \\
\beta &= \frac{\phi'^2 r^2}{4} - 1 - \Gamma'(r + \frac{\alpha' e^{\phi} \phi'}{2g^2}) + \frac{\alpha' e^{\phi}}{2g^2} \frac{w'^2}{e^2}
\end{aligned} \tag{19}$$

As a result e^{Λ}, as well as Λ', can be eliminated from equations (6), (7), (8) and (10). Choosing any three of them, since only three of them are independent, we obtain the system

$$\phi'' = -\frac{d_1}{d} \tag{20}$$

$$\Gamma'' = -\frac{d_2}{d} \tag{21}$$

$$w'' = -\frac{d_3}{d} \tag{22}$$

where d, d_1, d_2, d_3 are complicated functions of ϕ', Γ', w', ϕ, w, and r. Demanding that ϕ''_h and w''_h are finite, we obtain from (20) and (22) the constraints

$$\begin{aligned}
&\phi_h'^2 \, [\, 2e^{2\phi_h} C_h r_h \, (e^{2\phi_h} + e^{\phi_h} r_h^2 + r_h^4) - 8 e^{\phi_h} r_h^7 \,] + \\
&\phi_h' \, [\, e^{4\phi_h} C_h^2 + 4 e^{\phi_h} C_h r_h^2 \, (e^{2\phi_h} + 2e^{\phi_h} r_h^2 + r_h^4) - 16 r_h^8 \,] - \\
&e^{3\phi_h} C_h^2 r_h + 24 e^{2\phi_h} C_h r_h^3 + 8 e^{\phi_h} C_h r_h^5 - 48 e^{\phi_h} r_h^5 = 0
\end{aligned} \tag{23}$$

and

$$w_h' = -\frac{w_h \, (1 - w_h^2) \, (1 + e^{\phi_h} \phi_h' / 2r_h)}{r_h \, (1 - e^{\phi_h} C_h / 4 r_h^2)} \tag{24}$$

where $C_h = (1 - w_h^2)^2 / e^2$. Since α'/g^2 always multiplies e^{ϕ}, we may eliminate it from our calculations and restore it in the end. Substituting these two into eq.(21) gives

$$\Gamma'' = -\Gamma'^2 + \mathcal{O}(1) \Rightarrow \Gamma' = \frac{1}{r - r_h} + \mathcal{O}(1) \tag{25}$$

Expanding equation (18) near the horizon in powers of Γ', for $\delta = +1$ we obtain

$$e^{\Lambda} = \frac{2r^2 \, (2r + e^{\phi} \phi')}{4r^2 - e^{\phi} C} \, \Gamma' + \mathcal{O}(1) \tag{26}$$

while the choice $\delta = -1$ leads to $e^{\Lambda} = \mathcal{O}(1)$ which is not a black hole solution. Taking into account all the above, we may conclude that the unique black hole

solution with ϕ', ϕ'', w', w'' finite and $\Gamma' \to \infty$ near the horizon has the expansion

$$e^{\Gamma(r)} = \gamma_1(r-r_h) + \mathcal{O}(r-r_h)^2 \tag{27}$$

$$e^{-\Lambda(r)} = \lambda_1(r-r_h) + \mathcal{O}(r-r_h)^2 \tag{28}$$

$$\phi(r) = \phi_h + \phi'_h\,(r-r_h) + \frac{\phi''_h}{2}\,(r-r_h)^2 + \mathcal{O}(r-r_h)^3 \tag{29}$$

$$w(r) = w_h + w'_h\,(r-r_h) + \frac{w''_h}{2}\,(r-r_h)^2 + \mathcal{O}(r-r_h)^3 \tag{30}$$

where r_h, ϕ_h, w_h and γ_1 are free parameters while from eq.(26) we get

$$\lambda_1 = \frac{4r_h^2 - e^{\phi_h}\,C_h}{2r_h^2\,(2r_h + e^{\phi_h}\phi'_h)} \tag{31}$$

Note that the constraint (23) is an algebraic second order equation for ϕ'_h which has two real solutions, $\phi'_\pm$, only if its discriminant is positive. This ultimately gives

$$(C_h - z_+)(C_h - z_-) \geq 0 \tag{32}$$

where

$$z_\pm = \frac{4}{x^3}\,[-(2+3x+3x^2) \pm \sqrt{(2+3x+3x^2)^2 + 6x^2 - 1}\,] \tag{33}$$

having set $x = \alpha' e^{\phi_h}/g^2 r_h^2$. When $x \leq 1/\sqrt{6}$, or

$$\frac{\alpha' e^{\phi_h}}{g^2} \leq \frac{r_h^2}{\sqrt{6}} \tag{34}$$

the discriminant is always positive. This has the obvious interpretation that if the Gauss-Bonnet effective coupling $\alpha' e^{\phi_h}/g^2$ is smaller than the critical value $r_h^2/\sqrt{6}$, there is always a dilatonic black hole solution. The same constraint was derived in the case of Einstein-Dilaton-Gauss-Bonnet theory [4]. There seems to be, however, an additional possibility in the present case. The inequality (47) can be true either when $C_h \geq z_+ > z_-$ or when $C_h \leq z_- < z_+$. Since, however, $z_- < 0$ and $C_h > 0$, the latter case is impossible and only $C_h \geq z_+$ could, perhaps, be realized. Then, for $x \geq 1.11949$, the inequality $0 < z_+ < 1$ is true. Later we shall justify the choice $w_h < 1$, or equivalently $C_h < 1$. Thus, solutions could also be present if the effective Gauss-Bonnet coupling is in the region $\frac{\alpha' e^{\phi_h}}{g^2} \geq 1.11949\,r_h^2$. Nevertheless, no solutions were found in this new region.

Before we proceed to describe our numerical procedure and finally plot our solutions, let us compute analytically the temperature and entropy of the black hole. Introducing the Euclidean version of our metric

$$ds^2 = e^\Gamma d\tau^2 + e^\Lambda dr^2 + r^2(d\theta^2 + sin^2\theta d\varphi^2) \tag{35}$$

where τ is a periodic coordinate ranging over the interval $(0, \beta)$, we define the temperature as [8] [12]

$$T = \frac{\kappa}{2\pi} = \frac{1}{4\pi} \frac{1}{\sqrt{g_{tt} g_{rr}}} \left(\frac{d g_{tt}}{dr} \right)_{r=r_h} = \frac{\sqrt{\gamma_1 \lambda_1}}{4\pi} \tag{36}$$

where κ is the surface gravity of the black hole. On the other hand rearranging the equations of motion, we obtain

$$\frac{d}{dr} \left(e^{(\Gamma-\Lambda)/2} (\Gamma' - \phi')\, r^2 - \frac{\alpha' e^{\phi}}{g^2} e^{(\Gamma-\Lambda)/2} [(\phi' - \Gamma')\,(1 - e^{-\Lambda}) + e^{-\Lambda} \phi' \Gamma' r] \right) = 0 \tag{37}$$

The equation (37) is an identity. Integrating over the interval (r_h, r), we obtain

$$2M + D = \sqrt{\gamma_1 \lambda_1}\, (r_h^2 + \frac{\alpha' e^{\phi_h}}{g^2}) \tag{38}$$

which gives

$$T = T_{SW} \frac{r_h\,(2M + D)}{(r_h^2 + \frac{\alpha' e^{\phi_h}}{g^2})} \tag{39}$$

in terms of the temperature of a Schwarzschild black hole $T_{SW} = 1/8\pi M = 1/4\pi r_h$ and the asymptotic parameters M and D. Since T is a function of positive constants, it never vanishes. As a result, there is no mechanism to prevent the complete evaporation of the black hole. However, this is indeed the case only when the parameters r_h and ϕ_h obey the constraint (34). If, due to thermal evaporation, the parameters of the black hole stop to obey the above constraint, then this gravitational system can no longer be described by the concept of a regular black hole.

The entropy S can be derived from the free energy $F(\beta)$ of the system as

$$S = \beta\, [\, \frac{\partial(\beta F)}{\partial \beta} - F\,] \tag{40}$$

The free energy is defined as $F = I_E/\beta$, where I_E is the Euclidean version of the action

$$I_E = \int d^4x \sqrt{g}\, \mathcal{L} - \int d^4x \sqrt{g}\, K \tag{41}$$

where K is a suitably subtracted boundary term that comes from R and R^2_{GB}. We may perform the time and angular integration in (41) and by using the equations of motion the action takes the form

$$I_E = 2\pi\beta\, e^{(\Gamma-\Lambda)/2}\, [\, (\Gamma' - \phi')\, r^2 + \frac{\alpha' e^{\phi}}{g^2}\, (1 - e^{-\Lambda})\, (\Gamma' + \phi') - \frac{\alpha' e^{\phi}}{g^2}\, r e^{-\Lambda} \Gamma' \phi'\,]\, |_{r_h}^{\infty} \tag{42}$$

Substituting again the expansions near the horizon as well as near infinity and by making use of the definition (40), we obtain

$$S = \frac{A_H}{4}\,(1 + \frac{\alpha' e^{\phi_h}}{g^2 r_h^2}) \tag{43}$$

where $A_H = 4\pi r_h^2$ is the area of the event horizon and $S_{SW} = A_H/4$ is the Bekenstein-Hawking formula [13] for the entropy of the Schwarzschild black hole.

We note from eq.(1) that in the limit $\phi \to -\infty$ the effective coupling $\alpha' e^{\phi}/g^2$ vanishes. Then, the contribution of the Gauss-Bonnet term and the SU(2) potential to the equations of motion becomes trivial. We can show [4] that in this case the only acceptable solution is the standard Schwarzschild one with constant dilaton field, in agreement with the no-hair theorem. In this limit, we expect the temperature T and the entropy S of the coloured black hole to approach the corresponding Schwarzschild ones, T_{SW} and S_{SW}. As we can see from (39), since in this limit $D \to 0$, and (43), this is exactly what we obtain.

3. Numerical considerations and conclusions.

Taking into account the constraints (23),(24) and (31) we may conclude that the parameters of the problem are r_h, ϕ_h, w_h and γ_1. Note that the equations of motion (6)-(10) do not involve $\Gamma(r)$ but only $\Gamma'(r)$. The final integration determining $\Gamma(r)$ involves the integration constant γ_1 which will fixed by demanding asymptotic flatness through (14).

Considering the Yang-Mills equation at the horizon, we obtain

$$(e^{-\lambda})'_h w'_h + \frac{w_h\,(1-w_h^2)}{r_h^2} = 0 \Rightarrow sign\, w'_h = sign\, w_h\,(w_h^2 - 1) \tag{44}$$

According to this equation, if we choose the initial value w_h to be greater than one, then $w'_h > 0$, which means that $w\,(r)$ increases with r. If we want to constrain $w\,(r)$ at infinity by $w\,(\infty) = \pm 1$ then, a local maximum must occur at some point $r = r_s$. At this point we would have $w\,(r_s) > 1$ and $w'(r_s) = 0$. Using again the Yang-Mills equation, we obtain

$$sign\, w''(r_s) = sign\, w\,(r_s)\,[w^2(r_s) - 1] \tag{45}$$

If $w\,(r_s) > 1$, then $w''(r_s) > 0$, which means that we can have only a local minimum at r_s. As a result, the initial value of $w\,(r)$ must fall inside the interval $(-1, 1)$. Moreover, we observe that the equations of motion remain invariant under the transformation $w \to -w$. Thus, it would be sufficient to choose initial values of w_h in the interval $(0, 1)$.

The equations of motion (6)-(10) are invariant under the combined transformation

$$\phi \to \phi + \phi_0 \quad , \quad r \to r\, e^{\phi_0/2} \tag{46}$$

As a result, it is sufficient to vary only one of r_h and ϕ_h. Furthermore, we may use the above invariance to set a unique mass scale for all solutions by imposing the asymptotic condition $\phi_\infty = 0$. This requires a shift $\phi \to \phi - \phi_\infty$ accompanied by a rescaling $r \to re^{-\phi_\infty/2}$. Since the radial coordinate has been rescaled, the other two asymptotic parameters, M and D, are also rescaled according to the rule $M \to Me^{-\phi_\infty/2}$ and $D \to De^{-\phi_\infty/2}$. Similarly, the temperature T is also rescaled as $T \to Te^{\phi_\infty/2}$.

In order to perform the numerical integration it is convenient to set $\frac{\alpha'}{g^2} = e = 1$. We fix the value of the horizon at $r_h = 1$ and start by giving initial values to the remaining parameters ϕ_h and w_h. Starting from the expansions (27)-(30), at $r = r_h + \epsilon$ with $\epsilon \simeq O(10^{-8})$, we integrate the system (20)-(21) towards $r \to \infty$ using the fourth order Runge-Kutta method with an automatic step procedure and accuracy 10^{-8}. The integration stops when the flat space-time asymptotic limit (13)-(16) is reached. In the second allowed region of the Gauss-Bonnet effective coupling, defined by (47), and for the choices $\phi'_h = \phi'_+$ and $\phi'_h = \phi'_-$ we found solutions with regular and non-regular behaviour respectively near the horizon. However, in both cases the solutions lacked the right asymptotic behaviour (13)-(16) near infinity.

Making the choice $\phi'_h = \phi'_-$, which corresponds to the choice $\phi'_h = \phi'_+$ for the Einstein-Dilaton-Gauss-Bonnet theory, we were able to find regular asymptotically flat black hole solutions in the first allowed region defined by (34). For every initial value of the shooting parameter ϕ_h there is a discrete family of initial values of w_h which results into a discrete family of asymptotically flat black hole solutions characterized by the number n of zeros (nodes) of $w(r)$. We find that as $\phi_h \to -\infty$ the variation of the dilaton field $\phi(r)$ with r becomes weaker and the dilaton eventually behaves as a constant. The dilaton charge D moves towards zero and the horizon takes on its Schwarzschild value $r_h = 2M$. As we expect, the temperature T and the entropy S also take on the corresponding Schwarzschild ones. In the Table we display three sets of corresponding values of ϕ_h, w_h, and ϕ_∞. The displayed values of r_h, $2M$, D, and T are the rescaled ones corresponding to imposing the condition $\phi_\infty = 0$ at the end of our computation. Plots involving the dilaton field $\phi(r)$, for three different coloured black hole solutions, after the imposition of the asymptotic condition, are given in Figure 1. Figure 2 depicts the Yang-Mills function w of the $n = 1, 2, 3$ coloured black hole solutions as well as the metric function $g_{tt}(r)$. The dependence of the other metric function $g_{rr}(r)$ is presented in Figure 3.

TABLE I

Parameters of solutions for $r_h = 1$

ϕ_h	n	w_h	ϕ_∞	r_h	$2M$	D	T
-1.0	1	0.255948298	-1.43634	2.05067	2.28110	0.30503	0.03578
	2	0.042444284					
	3	0.006916638					
-2.5	1	0.262218961	-2.60493	3.67836	3.76249	0.14371	0.02123
	2	0.043590755					
	3	0.007101253					
-5.0	1	0.265648181	-5.00906	12.2378	12.2629	0.04275	0.00650
	2	0.044212029					
	3	0.007205569					

In the same region (34) and for the choice $\phi'_h = \phi'_+$ we found solutions which had the right asymptotic behaviour (13)-(16) near infinity and seemed to comprise a second group of regular black hole solutions. Examining more closely their behaviour near the horizon, they were found to possess a much more complicated structure. This structure is characterized mainly by the behaviour of the metric component g_{rr}, which is also given in Figure 3, while the other component, g_{tt}, exhibits a typical black hole behaviour. As we move from infinity towards the origin, the solution firstly reaches the value $r = r_s$, around which we may write the expansions

$$e^{-\Lambda} = \lambda_s + \lambda_2\,(r - r_s)^2 + ... \tag{47}$$

$$\Gamma' = \gamma_1 + \gamma_2\,(r - r_s) + ... \tag{48}$$

$$\phi = \phi_s + \phi'_s(r - r_s) + \frac{\phi''_s}{2}(r - r_s)^2 + ... \tag{49}$$

$$w = w_s + w'_s(r - r_s) + \frac{w''_s}{2}(r - r_s)^2 + ... \tag{50}$$

If we substitute these expansions into the equations of motion we obtain a set of constraints which determine γ_2, ϕ'_s, w'_s, ϕ''_s, and w''_s as functions of the free parameters λ_s, λ_2, γ_1, ϕ_s, and w_s. The *curvature invariant* $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ near r_s was found to be

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \lambda_s^2\,(\gamma_2 + \frac{\gamma_1^2}{2})^2 + \frac{2\gamma_1^2\lambda_s^2}{r_s^2} + \frac{4\,(1-\lambda_s)^2}{r_s^4} + \mathcal{O}(r - r_s) \tag{51}$$

We come to the conclusion that the surface $r = r_s$ is a regular one whose existence may be interpreted as an unsuccessful attempt of nature to form a regular horizon. The significance of this fact will become clear below. Moving beyond the value $r = r_s$, the solution eventually stops at the value $r = r_x$. The asymptotic behaviour

of the fields near r_x is

$$
\begin{aligned}
e^{-\Lambda} &= \lambda_x + \lambda_1 \sqrt{r - r_x} + ... && (52) \\
\Gamma' &= \gamma_1 + \gamma_2 \sqrt{r - r_x} + ... && (53) \\
\phi &= \phi_x + \phi'_x (r - r_x) + \phi''_x (r - r_s)^{3/2} + ... && (54) \\
w &= w_x + w'_x (r - r_x) + w''_x (r - r_s)^{3/2} + ... && (55)
\end{aligned}
$$

where again γ_2, ϕ'_x, w'_x, ϕ''_x and w''_x are given through the equations of motion as functions of the free parameters λ_x, λ_1, γ_1, ϕ_x and w_x. The *curvature invariant* $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ near r_x takes the form

$$
R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \left(\frac{\lambda_x^2}{4} (\gamma_2 + \frac{\gamma_1 \lambda_1}{2\lambda_x})^2 + \frac{\lambda_1^2}{2r_x^2} \right) \frac{1}{r - r_x} + \mathcal{O}(\frac{1}{\sqrt{r - r_x}}) \to \infty \qquad (56)
$$

This means that at the value $r = r_x$ the solution ends up in a pure scalar singularity. Since no regular horizon exists at some value $r = r_h > r_x$, we may conclude that a naked singularity has been formed at r_x.

For the critical value of ϕ_h, defined by (34), the two solutions ϕ'_- and ϕ'_+ coincide, which means that the branch of regular black hole solutions meets the branch of the solutions that describe a naked singularity. As long as the parameters r_h and ϕ_h of the solution obey the constraint (34), the only acceptable solution of the equations of motion is the regular black hole solution (27)-(30) with the flat asymptotic behaviour (13)-(16) near infinity. When r_h and ϕ_h become such - maybe due to thermal evaporation - that the constraint (34) is violated , the expansions (27)-(30) break down and the solution can no longer be described by the concept of the regular black hole. Then, the system shifts to the other branch of solutions, described above, corresponding to a naked singularity with exactly the same asymptotic behaviour at infinity.

Alexeyev et al.[5], by using a method based on integrating over an additional parameter, were able to examine the structure of the black hole solutions found in the Einstein-Dilaton-Gauss-Bonnet theory [4] inside the event horizon. According to their results, the solution under the regular horizon r_h exist only until the value $r = r_s$, where a pure scalar singularity exists. Another solution begins from r_s which exists only until the "singular horizon" r_x. When the Gauss Bonnet effective coupling becomes larger, the distance between r_s and r_h becomes smaller. Once the critical limit (34) is reached, $r_h = r_s = r_x$ and the internal structure vanishes. Since the Yang-Mills function w always resembles the behaviour of the dilaton field, we expect that the inclusion of the SU(2) Yang-Mills potential respects the above internal structure of the regular black hole. Actually, the similarity between the

internal structure of the black hole found by Alexeyev et al. and the structure of the solution describing a naked singularity found above is obvious. However, the interpretation of r_x and r_s is different: while in the first case they play the role of a "singular horizon" and a pure scalar singularity respectively inside the horizon, in the second case they stand for a scalar singularity and an unsuccessful event horizon respectively. As a result, we can make the further conjecture that as the effective coupling becomes larger the internal structure moves towards the horizon, for the critical value the points r_x, r_s and r_h merge, while for values beyond the critical point this structure penetrates the horizon and manifests itself as a naked singularity.

Note added in proof. While the present article was being completed, a related article titled "Dilatonic Black Holes with Gauss-Bonnet Term" by T. Torii, H. Yajima and K. Maeda, gr-qc/9606034 came into our attention.

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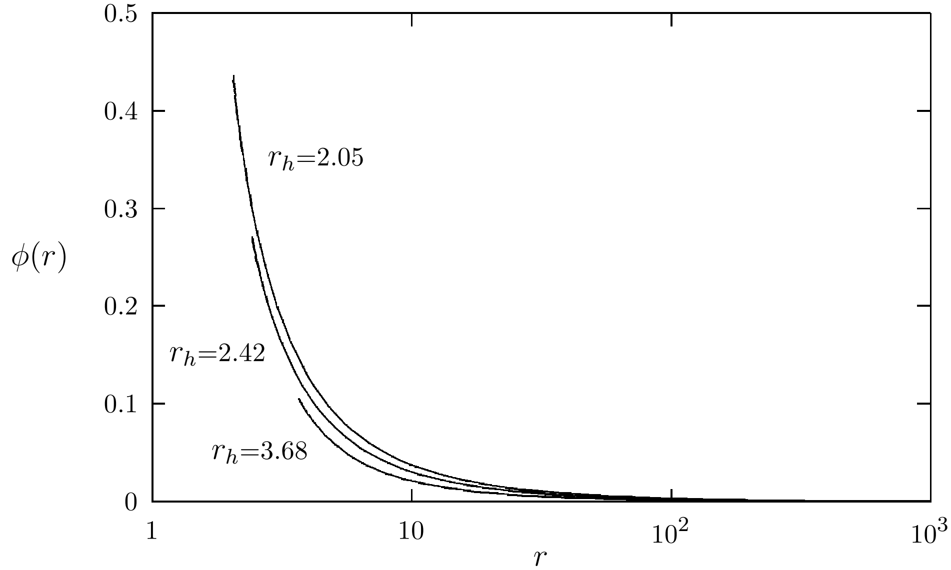

Figure 1: Dilaton field for coloured black hole solutions. Each curve corresponds to a different solution characterized by a different initial value of ϕ_h.

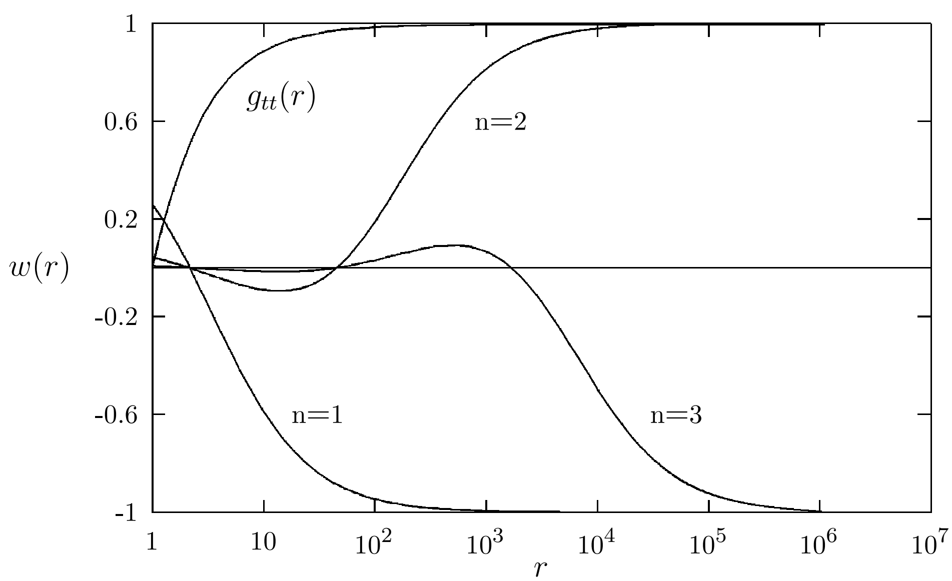

Figure 2: The Yang Mills function w of the $n = 1, 2, 3$ coloured black hole solutions as well as the metric component g_{tt} for $r_h = 1$ and $\phi_h = -1$.

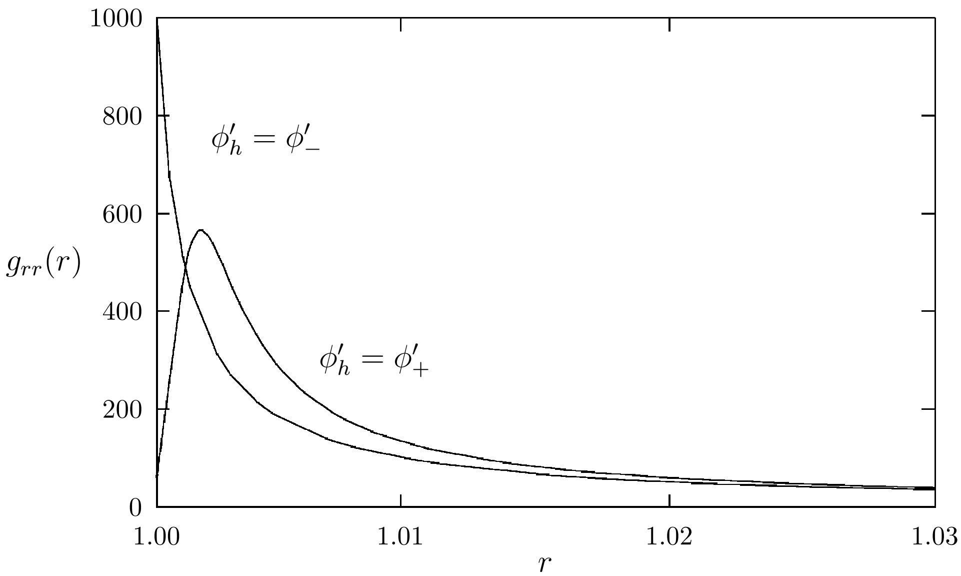

Figure 3: Dependence of the metric component g_{rr} for $r_h = 1$ and $\phi_h = -2$. While for $\phi'_h = \phi'_-$ g_{rr} moves towards an infinite value leading to the formation of an event horizon, for $\phi'_h = \phi'_+$ it reaches a maximum value at r_s and then declines leaving unshielded a scalar singularity.